Quadratic Equation

In mathematics, a quadratic equation is a polynomial equation of the second degree. The general form is

\[ ax^2 + bx + c = 0 \]

where \( a \neq 0 \). (For \( a = 0 \), the equation becomes a linear equation.)

The letters \( a \), \( b \), and \( c \) are called coefficients: the quadratic coefficient \( a \) is the coefficient of \( x^2 \), the linear coefficient \( b \) is the coefficient of \( x \), and \( c \) is the constant coefficient, also called the free term or constant term.

Example of Quadratic Equation

\[
\begin{align*}
2x^2 - 5 &= 0 \\
1 - 6x^2 &= 3 \\
6x + 3x^2 &= 0 \\
x^2 &= 0
\end{align*}
\]
Quadratic Equation

Solving Quadratic Equation
3 Methods:
- Factorisation
- Completing The Square
- Quadratic Formula

Factorisation

<table>
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<tr>
<th>Example</th>
<th>Solve $x^2 + 5x + 6 = 0$.</th>
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<tbody>
<tr>
<td>Answer</td>
<td>$x^2 + 5x + 6 = (x + 2)(x + 3)$</td>
</tr>
<tr>
<td></td>
<td>Set this equal to zero: $\frac{(x + 2)}{(x + 3)} = 0$</td>
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<tr>
<td></td>
<td>Solve each factor: $x + 2 = 0$ or $x + 3 = 0$</td>
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<tr>
<td></td>
<td>$x = -2$ or $x = -3$</td>
</tr>
<tr>
<td></td>
<td>The solution of $x^2 + 5x + 6 = 0$ is $x = -3, -2$</td>
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Completing The Square

<table>
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<th>e.g.1:</th>
<th>Solve the equation $x^2 - 2x - 5 = 0$.</th>
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<tbody>
<tr>
<td>Solution:</td>
<td>$x^2 - 2x - 5 = 0$</td>
</tr>
<tr>
<td></td>
<td>$(x^2 - 2x) - 5 = 0$</td>
</tr>
<tr>
<td></td>
<td>$[\frac{(x - 1)^2}{(x - 1)^2} - 1] = 5$</td>
</tr>
<tr>
<td></td>
<td>$(x - 1)^2 - 5 = 0$</td>
</tr>
<tr>
<td></td>
<td>$(x - 1)^2 - 6 = 0$</td>
</tr>
<tr>
<td></td>
<td>$(x - 1)^2 = 6$</td>
</tr>
<tr>
<td></td>
<td>$x - 1 = \pm\sqrt{6}$</td>
</tr>
<tr>
<td></td>
<td>$x = 1 \pm\sqrt{6}$</td>
</tr>
<tr>
<td></td>
<td>$x = 3.499@-1.499#$</td>
</tr>
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<th>e.g.2:</th>
<th>Solve the equation $x^2 - 6x - 10 = 0$.</th>
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<tbody>
<tr>
<td>Solution:</td>
<td>$x^2 - 6x - 10 = 0$</td>
</tr>
<tr>
<td></td>
<td>$(x^2 - 6x) - 10 = 0$</td>
</tr>
<tr>
<td></td>
<td>$[(x - 3)^2 - (3)^2] = 10$</td>
</tr>
<tr>
<td></td>
<td>$(x - 3)^2 = 19$</td>
</tr>
<tr>
<td></td>
<td>$x - 3 = \pm\sqrt{19}$</td>
</tr>
<tr>
<td></td>
<td>$x = 3 \pm\sqrt{19}$</td>
</tr>
<tr>
<td></td>
<td>$x = 7.359@-1.359#$</td>
</tr>
</tbody>
</table>
Quadratic Equation

Quadratic Formula

The formula to find roots, \[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \]

Prove that \[ x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a} \] if \[ ax^2 + bx + c = 0. \]

\[
ax^2 + bx + c = 0
\]

\[
(\div a)
\]

\[
\left( x^2 + \frac{b}{a}x + \frac{c}{a} \right) = 0
\]

\[
\left( x^2 + \frac{b}{2a}x + \frac{c}{2a} \right) = 0
\]

\[
\left( x^2 + \frac{b}{2a}x + \frac{c}{4a^2} \right) = 0
\]

e.g. 1:
Find the roots for the quadratic equation \[ 2x^2 - x - 7 = 0. \]

Solution:
\[
2x^2 - x - 7 = 0
\]
\[
a = 2, b = -1, c = -7
\]
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
x = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(2)(-7)}}{2(2)}
\]
\[
x = \frac{1 \pm \sqrt{57}}{1}
\]
\[
x = -1.637 @ 2.137
\]

e.g. 2:
Find the roots for the quadratic equation \[ 3x^2 - 3 = 4x. \]

Solution:
\[
3x^2 - 3 = 4x
\]
\[
a = 3, b = -4, c = -3
\]
\[
x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}
\]
\[
x = \frac{-(-4) \pm \sqrt{(-4)^2 - 4(3)(-3)}}{2(3)}
\]
\[
x = \frac{4 \pm \sqrt{52}}{6}
\]
\[
x = 1.869 @ -0.5351
\]
Quadratic Equation

Forming Quadratic Equation from Its Roots

<table>
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<tr>
<th>Topic 2.3 Quadratic Equations &amp; Functions: v 2.2 Form A Quadratic Equation, Solving Problem, &amp; Using the Formula</th>
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<tr>
<td>SOR &amp; POR: v 2.21 Given Roots</td>
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**e.g.1:** Form a quadratic equation whose roots are 1 and 2.

**Solution:**
\[
\begin{align*}
  x &= 1 \\
  x-1 &= 0 \\
  x-2 &= 0
\end{align*}
\]
\[
(x-1)(x-2) = 0 \\
 x^2-3x+2 = 0
\]

**Attention:** This question is using an inverse concept of factorisation.

**e.g.:**
\[
\begin{align*}
  x^2 - 3x + 2 &= 0 \\
  (x-2)(x-1) &= 0 \\
  x &= 1 @ 2
\end{align*}
\]

**e.g.2:** Form a quadratic equation whose roots are \( \frac{1}{3} \) and \(-\frac{1}{4} \).

**Solution:**
\[
\begin{align*}
  x &= \frac{1}{3} \\
  3x &= 1 \\
  3x-1 &= 0
\end{align*}
\]
\[
(3x-1)(4x+1) = 0 \\
12x^2-x-1 = 0
\]

**e.g.3:** Form a quadratic equation whose roots are 0 and -5.

**Solution:**
\[
\begin{align*}
  x &= 0 \\
  x-5 &= 0 \\
  x(x+5) &= 0
\end{align*}
\]
\[
x^2+5x = 0
\]

**e.g.4:** Form a quadratic equation whose roots are \( \frac{2}{3} \) and -1.

**Solution:**
\[
\begin{align*}
  x &= \frac{2}{3} \\
  3x &= 2 \\
  3x-2 &= 0
\end{align*}
\]
\[
(3x-2)(x+1) = 0 \\
3x^2+x-2 = 0
\]
Quadratic Equation

e.g. 2:
If \( \alpha \) and \( \beta \) are the roots for the quadratic equation \( x^2 + 3x + 5 = 0 \), Form a quadratic equation that has the following roots:

\[ a) \frac{\alpha + \beta}{2}, \frac{\beta - \alpha}{2} \]

\[ b) (2\alpha - 1), (2\beta - 1) \]

(a) Solution:
Let the new equation as \( x^2 + Bx + C = 0 \),
\[ a = 1, b = B, c = C \]

New Roots: \( \frac{\alpha + \beta}{2}, \frac{\beta - \alpha}{2} \)

\[ SOR: \frac{\alpha + \beta}{2} = \frac{-B}{1} \]
\[ \frac{\alpha + \beta}{2} = \frac{-B}{2} \]
\[ \frac{\alpha + \beta}{2} = \frac{-B}{2} \]

\[ (\alpha + \beta) = -B \]
\[ \frac{\alpha + \beta}{2} = \frac{-3}{2} \]
\[ B = \frac{3}{2} \]

\[ x^2 + \frac{3}{2}x + \frac{5}{4} = 0 \]
\[ (a) \]
\[ 4x^2 + 6x + 5 = 0 \]

::: The quadratic equation that has the roots \( \frac{\alpha + \beta}{2} \)
is \( 4x^2 + 6x + 5 = 0 \)

(b) Solution:
Let the new equation as \( x^2 + Bx + C = 0 \),
\[ a = 1, b = B, c = C \]

New Roots: \( (2\alpha - 1), (2\beta - 1) \)

\[ SOR: (2\alpha - 1) + (2\beta - 1) = \frac{-B}{1} \]
\[ \frac{(2\alpha + \beta) - 2}{2} = \frac{-B}{2} \]
\[ (2\alpha + \beta) - 2 \]
\[ (2\alpha - 1) + (2\beta - 1) \]
\[ = \frac{-B}{2} \]

\[ 2(\alpha + \beta) - 2 = -B \]
\[ 2(\alpha + \beta) - 2 = -B \]
\[ = \frac{5}{2} \]

\[ B = \frac{5}{4} \]

\[ C = \frac{5}{4} \]

\[ \therefore \text{The quadratic equation that has the roots } (2\alpha - 1), (2\beta - 1) \text{ is } x^2 + 8x + 27 = 0 \]

::: The quadratic equation can not has fraction

e.g. 3:
If \( \alpha \) and \( \beta \) are the roots for the quadratic equation \( x^2 + 2x - 5 = 0 \), Form a quadratic equation that has the roots:

**a) \( \alpha^2, \beta^2 \)**

**b) \( \frac{1}{\alpha^2 + \beta^2} \)**

**c) \( \frac{\alpha}{\beta^2}, \frac{\beta}{\alpha^2} \)**

Solution:
***(a)***

Let the new equation as \( x^2 + Bx + C = 0 \),
\[ a = 1, b = B, c = C \]

New Roots: \( \alpha^2, \beta^2 \)

\[ (\alpha + \beta)^2 = \alpha^2 + 2\alpha\beta + \beta^2 \]

\[ \alpha^2 + \beta^2 = (\alpha + \beta)^2 - 2\alpha\beta \]

Use this formula if has \( \alpha^3 + \beta^3 \)

\[ SOR: \frac{\alpha^2 + \beta^2}{1} = \frac{-B}{1} \]
\[ \frac{\alpha^2 + \beta^2}{1} = \frac{-B}{1} \]
\[ (\alpha + \beta)^2 - 2\alpha\beta = -B \]
\[ (-2)^2 - 2(-5) = -B \]
\[ -B = 14 \]
\[ B = 14 \]

\[ x^2 - 14x + 25 = 0 \]

::: The quadratic equation that has the roots \( \alpha^2, \beta^2 \)
is \( x^2 - 14x + 25 = 0 \)
Quadratic Equation

(b) New Roots: \( \frac{1}{\alpha}, \frac{1}{\beta} \)

\[ \alpha + \beta = -2, \alpha \beta = -5 \]

Let the quadratic equation as \( x^2 + Bx + C = 0 \).

\[
\begin{align*}
\text{SOR:} & \quad \frac{1}{\alpha} + \frac{1}{\beta} = -\frac{B}{1} \\
& \quad \frac{\alpha + \beta}{\alpha \beta} = -B
\end{align*}
\]

\[ \alpha + \beta = -B \]

\[ \frac{1}{\alpha \beta} = C \]

\[ \frac{-2}{-5} = -B \]

\[ \frac{-5}{-5} = C \]

\[ -B = \frac{2}{5} \]

\[ C = -\frac{1}{5} \]

\[ B = \frac{2}{5} \]

\[ x^2 - \frac{2}{5}x - \frac{1}{5} = 0 \]

\[ 5x^2 - 2x - 1 = 0 \]

**: The quadratic equation that has the roots \( \frac{1}{\alpha}, \frac{1}{\beta} \) is \( 5x^2 - 2x - 1 = 0 \)**

(c) New Roots: \( \frac{\alpha}{\beta}, \frac{\beta}{\alpha} \)

\[ \alpha + \beta = -2, \alpha \beta = -5 \]

Let the quadratic equation as \( x^2 + Bx + C = 0 \).

\[
\begin{align*}
\text{SOR:} & \quad \frac{\alpha}{\beta} + \frac{\beta}{\alpha} = -\frac{B}{1} \\
& \quad \frac{\alpha^2 + \beta^2}{\alpha \beta} = -B
\end{align*}
\]

\[ \frac{\alpha^2 + \beta^2}{\alpha \beta} = -B \]

\[ \frac{(-5)}{-5} = C \]

\[ -B = \frac{2}{3} = \frac{-5}{3} \]

\[ B = \frac{14}{5} \]

\[ x^2 + \frac{14}{5}x + 1 = 0 \]

\[ 5x^2 + 14x + 5 = 0 \]

**: The quadratic equation that has the roots \( \frac{\alpha}{\beta}, \frac{\beta}{\alpha} \) is \( 5x^2 + 14x + 5 = 0 \)**

---

**Trick Questions**

1. If \( \alpha \) and \( \beta \) are the roots for the quadratic equation \( x^2 - 2x - 3 = 0 \), Form a quadratic equation that has the roots \( \frac{\alpha}{\beta^2} \) and \( \frac{\beta}{\alpha^2} \). \( \alpha > \beta \rightarrow \) can calculate the \( \alpha \) and \( \beta \).

**Solution:**

\[ x^2 - 2x - 3 = 0 \]

\[ (x-3)(x+1) = 0 \]

\[ x = -1 @ 3 \]

Given that \( \alpha > \beta, \therefore \alpha = 3, \beta = -1 \)

\[ x = \frac{\alpha}{\beta^2} \]

\[ x = \frac{\beta}{\alpha^2} \]

\[ x = \frac{3}{(-1)^2} \]

\[ x = \frac{-1}{(3)^2} \]

\[ x = 3 \]

\[ x = -\frac{1}{9} \]

\[ x-3 = 0 \]

\[ 9x = -1 \]

\[ 9x+1 = 0 \]

\[ (x-3)(9x+1) = 0 \]

**: \( x^2 - 26x - 3 = 0 \)**
Quadratic Equation

2. If $\alpha$ and $\beta$ are the roots for the quadratic equation $x^2 - x - 6 = 0$, Form a quadratic equation that has the roots $\alpha$ and $\alpha > \beta$.

Solution:

$x^2 - x - 6 = 0$

\[(x-3)(x+2) = 0\]

\[x = -2, 3\]

Given that $\alpha > \beta, \therefore \alpha = 3, \beta = -2$

\[x = \alpha \quad x = \frac{1}{\beta} \]

\[x = 3 \quad x = \frac{1}{-2} \]

\[2x = -1 \quad 2x + 1 = 0\]

\[(x-3)(2x+1) = 0 \]

\[\therefore 2x^2 - 5x - 3 = 0\]

3. The roots for a quadratic equation $2x^2 - (k-2)x - 3k = 0$ are $p$ and $q$. If $q = 1 - p$, Calculate the possible value(s) of $k$, $p$ and $q$.

Solution:

\[2x^2 - (k-2)x - 3k = 0\]

$a = 2, b = -(k-2), c = -3k$

roots: $p, q$

\[p + q = -(k-2) \]

\[pq = \frac{-3k}{2} \]

But $q = 1 - p$,

\[p + q = 1 \]

From 1 and 3,

\[\frac{k-2}{2} = 1 \]

\[k = 4\]

Substitute $k$ and 4,

$2x^2 - (4 - 2)x - 3(4) = 0$

$2x^2 - 2x - 12 = 0$

$x^2 - x - 6 = 0$

\[(x-3)(x+2) = 0\]

$x = -2, 3$

If $p = -2$, then $q = 3$ if $p = 3$, then $q = -2$
Quadratic Equation

4. If one of the roots for the quadratic equation \(x^2 - ax + b = 0\) is twice the other root. Prove that \(9b = 2a^2\).

**Solution:**
roots: \(\alpha, 2\alpha\)

**SOR:**
\[
\alpha + 2\alpha = -\frac{(-a)}{1} \\
3\alpha = a \\
\alpha = \frac{a}{3} \quad \text{.................(1)}
\]

**POR:**
\[
\alpha \times 2\alpha = \frac{(b)}{1} \\
2\alpha^2 = b \\
\alpha^2 = \frac{b}{2} \quad \text{.................(2)}
\]

Substitute \(\alpha\) into (2),
\[
2\left(\frac{\alpha^2}{3}\right) = b \\
2\left(\frac{a^2}{9}\right) = b \\
2a^2 = 9b \quad \#
\]

**Nature of Roots of a Quadratic Equation**
The expression \(b^2 - 4ac\) in the general formula is called the **discriminant** of the equation, as it determines the type of roots that the equation has.

\[
\begin{align*}
b^2 - 4ac &> 0 \Leftrightarrow \text{two real and distinct roots} \\
b^2 - 4ac &= 0 \Leftrightarrow \text{two real and equal roots} \\
b^2 - 4ac &< 0 \Leftrightarrow \text{no real roots} \\
b^2 - 4ac &\geq 0 \Leftrightarrow \text{the roots are real}
\end{align*}
\]

**e.g. 1:**
Find the range of values of \(k\) for which the equation \(2x^2 + 5x + 3 - k = 0\) has two real distinct roots.

\[
\begin{align*}
b^2 - 4ac &> 0 \\
(5)^2 - 4(2)(3-k) &> 0 \\
25 - 24 + 8k &> 0 \\
1 + 8k &> 0 \\
8k &> -1 \\
k &> -\frac{1}{8}
\end{align*}
\]

**e.g. 2:**
The roots of \(3x^2 + kx + 12 = 0\) are equal. Find \(k\).

\[
\begin{align*}
b^2 - 4ac &= 0 \\
(k)^2 - 4(3)(12) &= 0 \\
k^2 - 144 &= 0 \\
k^2 &= 144 \\
k &= \pm\sqrt{144} \\
k &= \pm12
\end{align*}
\]

**e.g. 3:**
Find the range of values of \(p\) for which the equation \(x^2 - 2px + p^2 + 5p - 6 = 0\) has no real roots.

\[
\begin{align*}
b^2 - 4ac &< 0 \\
(-2p)^2 - 4(1)(p^2 + 5p - 6) &< 0 \\
4p^2 - 4p^2 - 20p + 24 &< 0 \\
-20p + 24 &< 0 \\
-20p &< -24 \\
p &> \frac{24}{20} \\
p &> \frac{6}{5}
\end{align*}
\]
Quadratic Equation

**e.g. 4:**
Show that the equation $a^2x^2 + 3ax + 2 = 0$
always has real roots.

\[ b^2 - 4ac \\
= (3a)^2 - 4(a^2)(2) \\
= 9a^2 - 8a^2 \\
= a^2 \]

$a^2 > 0$ for all values of $a$. Therefore

\[ b^2 - 4ac > 0 \]

Proven that

$a^2x^2 + 3ax + 2 = 0$ always has real roots.