Chapter 6 Numerical Methods

Oct/Nov 2002

7

The diagram shows a curved rod $AB$ of length 100 cm which forms an arc of a circle. The end points $A$ and $B$ of the rod are 99 cm apart. The circle has radius $r$ cm and the arc $AB$ subtends an angle of $2\alpha$ radians at $O$, the centre of the circle.

(i) Show that $\alpha$ satisfies the equation $\frac{99}{100} x = \sin x$. [3]

(ii) Given that this equation has exactly one root in the interval $0 < x < \frac{1}{2}\pi$, verify by calculation that this root lies between 0.1 and 0.5. [2]

(iii) Show that if the sequence of values given by the iterative formula

$$x_{n+1} = 50 \sin x_n - 48.5 x_n$$

converges, then it converges to a root of the equation in part (i). [2]

(iv) Use this iterative formula, with initial value $x_1 = 0.25$, to find $\alpha$ correct to 3 decimal places, showing the result of each iteration. [2]

May/June 2003

8 The equation of a curve is $y = \ln x + \frac{2}{x}$, where $x > 0$.

(i) Find the coordinates of the stationary point of the curve and determine whether it is a maximum or a minimum point. [5]

(ii) The sequence of values given by the iterative formula

$$x_{n+1} = \frac{2}{3 - \ln x_n},$$

with initial value $x_1 = 1$, converges to $\alpha$. State an equation satisfied by $\alpha$, and hence show that $\alpha$ is the $x$-coordinate of a point on the curve where $y = 3$. [2]

(iii) Use this iterative formula to find $\alpha$ correct to 2 decimal places, showing the result of each iteration. [3]
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Oct/Nov 2003

5 (i) By sketching suitable graphs, show that the equation

\[ \sec x = 3 - x^2 \]

has exactly one root in the interval \( 0 < x < \frac{1}{2} \pi \). \[ 2 \]

(ii) Show that, if a sequence of values given by the iterative formula

\[ x_{n+1} = \cos^{-1}\left( \frac{1}{3 - x_n^2} \right) \]

converges, then it converges to a root of the equation given in part (i). \[ 2 \]

(iii) Use this iterative formula, with initial value \( x_1 = 1 \), to determine the root in the interval \( 0 < x < \frac{1}{2} \pi \) correct to 2 decimal places, showing the result of each iteration. \[ 3 \]

May/June 2004

7 (i) The equation \( x^3 + x + 1 = 0 \) has one real root. Show by calculation that this root lies between \(-1\) and 0. \[ 2 \]

(ii) Show that, if a sequence of values given by the iterative formula

\[ x_{n+1} = \frac{2x_n^3 - 1}{3x_n^2 + 1} \]

converges, then it converges to the root of the equation given in part (i). \[ 2 \]

(iii) Use this iterative formula, with initial value \( x_1 = -0.5 \), to determine the root correct to 2 decimal places, showing the result of each iteration. \[ 3 \]

Oct/Nov 2004

5

The diagram shows a sector \( OAB \) of a circle with centre \( O \) and radius \( r \). The angle \( AOB \) is \( \alpha \) radians, where \( 0 < \alpha < \frac{1}{2} \pi \). The point \( N \) on \( OA \) is such that \( BN \) is perpendicular to \( OA \). The area of the triangle \( ONB \) is half the area of the sector \( OAB \).

(i) Show that \( \alpha \) satisfies the equation \( \sin 2\alpha = x \). \[ 3 \]

(ii) By sketching a suitable pair of graphs, show that this equation has exactly one root in the interval \( 0 < x < \frac{1}{2} \pi \). \[ 2 \]

(iii) Use the iterative formula

\[ x_{n+1} = \sin(2x_n) \]

with initial value \( x_1 = 1 \), to find \( \alpha \) correct to 2 decimal places, showing the result of each iteration. \[ 3 \]
The diagram shows a sketch of the curve \( y = \frac{1}{1 + x^2} \) for values of \( x \) from \(-0.6\) to \(0.6\).

(i) Use the trapezium rule, with two intervals, to estimate the value of

\[
\int_{-0.6}^{0.6} \frac{1}{1 + x^2} \, dx,
\]

giving your answer correct to 2 decimal places. \[3\]

(ii) Explain, with reference to the diagram, why the trapezium rule may be expected to give a good approximation to the true value of the integral in this case. \[1\]

7 (i) By sketching a suitable pair of graphs, show that the equation

\[ \cosec x = \frac{1}{2}x + 1, \]

where \( x \) is in radians, has a root in the interval \(0 < x < \frac{1}{2}\pi\). \[2\]

(ii) Verify, by calculation, that this root lies between 0.5 and 1. \[2\]

(iii) Show that this root also satisfies the equation

\[ x = \sin^{-1}\left(\frac{2}{x + 2}\right) \]

\[1\]

(iv) Use the iterative formula

\[ x_{n+1} = \sin^{-1}\left(\frac{2}{x_n + 2}\right) \]

with initial value \( x_1 = 0.75 \), to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. \[3\]
4 The equation \( x^3 - x - 3 = 0 \) has one real root, \( \alpha \).

(i) Show that \( \alpha \) lies between 1 and 2. 

Two iterative formulae derived from this equation are as follows:

\[
x_{n+1} = x_n^3 - 3, \quad (A)
\]

\[
x_{n+1} = (x_n + 3)^{\frac{1}{3}}, \quad (B)
\]

Each formula is used with initial value \( x_1 = 1.5 \).

(ii) Show that one of these formulae produces a sequence which fails to converge, and use the other formula to calculate \( \alpha \) correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

May/June 2006

6 (i) By sketching a suitable pair of graphs, show that the equation

\[ 2 \cot x = 1 + e^x, \]

where \( x \) is in radians, has only one root in the interval \( 0 < x < \frac{1}{2} \pi \).

(ii) Verify by calculation that this root lies between 0.5 and 1.0.

(iii) Show that this root also satisfies the equation

\[ x = \tan^{-1}\left( \frac{2}{1 + e^x} \right). \]

(iv) Use the iterative formula

\[ x_{n+1} = \tan^{-1}\left( \frac{2}{1 + e^x} \right), \]

with initial value \( x_1 = 0.7 \), to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.
The diagram shows the curve \( y = x \cos 2x \) for \( 0 \leq x \leq \frac{1}{4}\pi \). The point \( M \) is a maximum point.

(i) Show that the \( x \)-coordinate of \( M \) satisfies the equation \( 1 = 2x \tan 2x \). \[3\]

(ii) The equation in part (i) can be rearranged in the form \( x = \frac{1}{2} \tan^{-1}\left(\frac{1}{2\tan^2 x}\right) \). Use the iterative formula

\[
x_{n+1} = \frac{1}{2} \tan^{-1}\left(\frac{1}{2\tan^2 x_n}\right),
\]

with initial value \( x_1 = 0.4 \), to calculate the \( x \)-coordinate of \( M \) correct to 2 decimal places. Give the result of each iteration to 4 decimal places. \[3\]

(iii) Use integration by parts to find the exact area of the region enclosed between the curve and the \( x \)-axis from 0 to \( \frac{1}{4}\pi \). \[5\]

May/June 2007

6

The diagram shows a sector \( AOB \) of a circle with centre \( O \) and radius \( r \). The angle \( AOB \) is \( \alpha \) radians, where \( 0 < \alpha < \pi \). The area of triangle \( AOB \) is half the area of the sector.

(i) Show that \( \alpha \) satisfies the equation \( x = 2 \sin x \). \[2\]

(ii) Verify by calculation that \( \alpha \) lies between \( \frac{1}{2}\pi \) and \( \frac{3}{2}\pi \). \[2\]

(iii) Show that, if a sequence of values given by the iterative formula

\[
x_{n+1} = \frac{1}{3}(x_n + 4 \sin x_n)
\]

converges, then it converges to a root of the equation in part (i). \[2\]

(iv) Use this iterative formula, with initial value \( x_1 = 1.8 \), to find \( \alpha \) correct to 2 decimal places. Give the result of each iteration to 4 decimal places. \[3\]
6 (i) By sketching a suitable pair of graphs, show that the equation
\[ 2 - x = \ln x \]
has only one root. \[2\]

(ii) Verify by calculation that this root lies between 1.4 and 1.7. \[2\]

(iii) Show that this root also satisfies the equation
\[ x = \frac{1}{3}(4 + x - 2 \ln x). \] \[1\]

(iv) Use the iterative formula
\[ x_{n+1} = \frac{1}{3}(4 + x_n - 2 \ln x_n), \]
with initial value \( x_1 = 1.5 \), to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. \[3\]

May/June 2008

3

In the diagram, \( ABCD \) is a rectangle with \( AB = 3a \) and \( AD = a \). A circular arc, with centre \( A \) and radius \( r \), joins points \( M \) and \( N \) on \( AB \) and \( CD \) respectively. The angle \( MAN \) is \( x \) radians. The perimeter of the sector \( AMN \) is equal to half the perimeter of the rectangle.

(i) Show that \( x \) satisfies the equation
\[ \sin x = \frac{1}{4}(2 + x). \] \[3\]

(ii) This equation has only one root in the interval \( 0 < x < \frac{1}{2}\pi \). Use the iterative formula
\[ x_{n+1} = \sin^{-1}\left(\frac{2 + x_n}{4}\right), \]
with initial value \( x_1 = 0.8 \), to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. \[3\]
The diagram shows the curve \( y = \sqrt{1 + 2 \tan^2 x} \) for \( 0 \leq x \leq \frac{1}{4} \pi \).

(i) Use the trapezium rule with three intervals to estimate the value of
\[
\int_{0}^{\frac{1}{4} \pi} \sqrt{1 + 2 \tan^2 x} \, dx,
\]
giving your answer correct to 2 decimal places. [3]

(ii) The estimate found in part (i) is denoted by \( E \). Explain, without further calculation, whether another estimate found using the trapezium rule with six intervals would be greater than \( E \) or less than \( E \). [1]

4 The equation \( x^3 - 2x - 2 = 0 \) has one real root.

(i) Show by calculation that this root lies between \( x = 1 \) and \( x = 2 \). [2]

(ii) Prove that, if a sequence of values given by the iterative formula
\[
x_{n+1} = \frac{2x_n^3 + 2}{3x_n^2 - 2},
\]
converges, then it converges to this root. [2]

(iii) Use this iterative formula to calculate the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places. [3]

Oct/Nov 2009/31

3 The sequence of values given by the iterative formula
\[
x_{n+1} = \frac{3x_n}{4} + \frac{15}{x_n^3},
\]
with initial value \( x_1 = 3 \), converges to \( \alpha \).

(i) Use this iterative formula to find \( \alpha \) correct to 2 decimal places, giving the result of each iteration to 4 decimal places. [3]

(ii) State an equation satisfied by \( \alpha \) and hence find the exact value of \( \alpha \). [2]
The equation \( x^3 - 8x - 13 = 0 \) has one real root.

(i) Find the two consecutive integers between which this root lies.  [2]

(ii) Use the iterative formula

\[
x_{n+1} = (8x_n + 13)^{\frac{1}{3}}
\]


to determine this root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.  [3]

The diagram shows a semicircle \( ACB \) with centre \( O \) and radius \( r \). The angle \( BOC \) is \( x \) radians. The area of the shaded segment is a quarter of the area of the semicircle.

(i) Show that \( x \) satisfies the equation

\[
x = \frac{3}{2} \pi - \sin x.
\]

(ii) This equation has one root. Verify by calculation that the root lies between 1.3 and 1.5.  [2]

(iii) Use the iterative formula

\[
x_{n+1} = \frac{3}{2} \pi - \sin x_n
\]


to determine the root correct to 2 decimal places. Give the result of each iteration to 4 decimal places.  [3]
The diagram shows the curve \( y = \frac{\sin x}{x} \) for \( 0 < x \leq 2\pi \), and its minimum point \( M \).

(i) Show that the \( x \)-coordinate of \( M \) satisfies the equation
\[
x = \tan x.
\]

(ii) The iterative formula
\[
x_{n+1} = \tan^{-1}(x_n) + \pi
\]
can be used to determine the \( x \)-coordinate of \( M \). Use this formula to determine the \( x \)-coordinate of \( M \) correct to 2 decimal places. Give the result of each iteration to 4 decimal places.

May/Jun 2010/33

6 The curve \( y = \frac{\ln x}{x+1} \) has one stationary point.

(i) Show that the \( x \)-coordinate of this point satisfies the equation
\[
x = \frac{x + 1}{\ln x},
\]
and that this \( x \)-coordinate lies between 3 and 4.

(ii) Use the iterative formula
\[
x_{n+1} = \frac{x_n + 1}{\ln x_n}
\]
to determine the \( x \)-coordinate correct to 2 decimal places. Give the result of each iteration to 4 decimal places.