Chapter 9 Complex numbers

May/June 2002

9 The complex number \(1 + i \sqrt{3}\) is denoted by \(u\).

(i) Express \(u\) in the form \(r(\cos \theta + i \sin \theta)\), where \(r > 0\) and \(-\pi < \theta \leq \pi\). Hence, or otherwise, find the modulus and argument of \(u^2\) and \(u^3\). \[5\]

(ii) Show that \(u\) is a root of the equation \(z^2 - 2z + 4 = 0\), and state the other root of this equation. \[2\]

(iii) Sketch an Argand diagram showing the points representing the complex numbers \(i\) and \(u\). Shade the region whose points represent every complex number \(z\) satisfying both the inequalities \[|z - i| \leq 1\quad \text{and} \quad \arg z > \arg u.\] \[4\]

Oct/Nov 2002

8 (a) Find the two square roots of the complex number \(-3 + 4i\), giving your answers in the form \(x + iy\), where \(x\) and \(y\) are real. \[5\]

(b) The complex number \(z\) is given by \[z = \frac{-1 + 3i}{2 + i}.\]

(i) Express \(z\) in the form \(x + iy\), where \(x\) and \(y\) are real. \[2\]

(ii) Show on a sketch of an Argand diagram, with origin \(O\), the points \(A\), \(B\) and \(C\) representing the complex numbers \(-1 + 3i\), \(2 + i\) and \(z\) respectively. \[1\]

(iii) State an equation relating the lengths \(OA\), \(OB\) and \(OC\). \[1\]

May/June 2003

5 The complex number \(2i\) is denoted by \(u\). The complex number with modulus 1 and argument \(\frac{2}{3}\pi\) is denoted by \(w\).

(i) Find in the form \(x + iy\), where \(x\) and \(y\) are real, the complex numbers \(w\), \(uw\) and \(\frac{u}{w}\). \[4\]

(ii) Sketch an Argand diagram showing the points \(U\), \(A\) and \(B\) representing the complex numbers \(u\), \(uw\) and \(\frac{u}{w}\) respectively. \[2\]

(iii) Prove that triangle \(UAB\) is equilateral. \[2\]

Oct/Nov 2003

7 The complex number \(u\) is given by \(u = \frac{7 + 4i}{3 - 2i}\).

(i) Express \(u\) in the form \(x + iy\), where \(x\) and \(y\) are real. \[3\]

(ii) Sketch an Argand diagram showing the point representing the complex number \(u\). Show on the same diagram the locus of the complex number \(z\) such that \(|z - u| = 2\). \[3\]

(iii) Find the greatest value of \(\arg z\) for points on this locus. \[3\]
May/June 2004

8  
(i) Find the roots of the equation \(z^2 - z + 1 = 0\), giving your answers in the form \(x + iy\), where \(x\) and \(y\) are real. \([2]\)

(ii) Obtain the modulus and argument of each root. \([3]\)

(iii) Show that each root also satisfies the equation \(z^3 = -1\). \([2]\)

Oct/June 2004

6  
The complex numbers \(1 + 3i\) and \(4 + 2i\) are denoted by \(u\) and \(v\) respectively.

(i) Find, in the form \(x + iy\), where \(x\) and \(y\) are real, the complex numbers \(u - v\) and \(\frac{u}{v}\). \([3]\)

(ii) State the argument of \(\frac{u}{v}\). \([1]\)

In an Argand diagram, with origin \(O\), the points \(A\), \(B\) and \(C\) represent the numbers \(u\), \(v\) and \(u - v\) respectively.

(iii) State fully the geometrical relationship between \(OC\) and \(BA\). \([2]\)

(iv) Prove that angle \(AOB = \frac{1}{3}\pi\) radians. \([2]\)

May/June 2005

3  
(i) Solve the equation \(z^2 - 2iz - 5 = 0\), giving your answers in the form \(x + iy\) where \(x\) and \(y\) are real. \([3]\)

(ii) Find the modulus and argument of each root. \([3]\)

(iii) Sketch an Argand diagram showing the points representing the roots. \([1]\)

Oct/Nov 2005

7  
The equation \(2x^3 + x^2 + 25 = 0\) has one real root and two complex roots.

(i) Verify that \(1 + 2i\) is one of the complex roots. \([3]\)

(ii) Write down the other complex root of the equation. \([1]\)

(iii) Sketch an Argand diagram showing the point representing the complex number \(1 + 2i\). Show on the same diagram the set of points representing the complex numbers \(z\) which satisfy

\[|z| = |z - 1 - 2i|\]. \([4]\)
7 The complex number $2 + i$ is denoted by $u$. Its complex conjugate is denoted by $u^*$. 

(i) Show, on a sketch of an Argand diagram with origin $O$, the points $A$, $B$ and $C$ representing the complex numbers $u$, $u^*$ and $u + u^*$ respectively. Describe in geometrical terms the relationship between the four points $O$, $A$, $B$ and $C$. [4]

(ii) Express $\frac{u}{u^*}$ in the form $x + iy$, where $x$ and $y$ are real. [3]

(iii) By considering the argument of $\frac{u}{u^*}$, or otherwise, prove that 

$$\tan^{-1}(\frac{4}{3}) = 2 \tan^{-1}(\frac{1}{2}).$$

[2]

Oct/Nov 2006

9 The complex number $u$ is given by 

$$u = \frac{3 + i}{2 - i}.$$ 

(i) Express $u$ in the form $x + iy$, where $x$ and $y$ are real. [3]

(ii) Find the modulus and argument of $u$. [2]

(iii) Sketch an Argand diagram showing the point representing the complex number $u$. Show on the same diagram the locus of the point representing the complex number $z$ such that $|z - u| = 1$. [3]

(iv) Using your diagram, calculate the least value of $|z|$ for points on this locus. [2]

May/June 2007

8 The complex number $\frac{2}{-1 + i}$ is denoted by $u$.

(i) Find the modulus and argument of $u$ and $u^2$. [6]

(ii) Sketch an Argand diagram showing the points representing the complex numbers $u$ and $u^2$. Shade the region whose points represent the complex numbers $z$ which satisfy both the inequalities $|z| < 2$ and $|z - u^2| < |z - u|$. [4]

Oct/Nov 2007

8 (a) The complex number $z$ is given by $z = \frac{4 - 3i}{1 - 2i}$.

(i) Express $z$ in the form $x + iy$, where $x$ and $y$ are real. [2]

(ii) Find the modulus and argument of $z$. [2]

(b) Find the two square roots of the complex number $5 - 12i$, giving your answers in the form $x + iy$, where $x$ and $y$ are real. [6]
The variable complex number $z$ is given by

$$z = 2 \cos \theta + i(1 - 2 \sin \theta),$$

where $\theta$ takes all values in the interval $-\pi < \theta \leq \pi$.

(i) Show that $|z - i| = 2$, for all values of $\theta$. Hence sketch, in an Argand diagram, the locus of the point representing $z$. [3]

(ii) Prove that the real part of $\frac{1}{z + 2 - i}$ is constant for $-\pi < \theta < \pi$. [4]

Oct/Nov 2008

The complex number $w$ is given by $w = \frac{1}{2} + i\sqrt{3}$. [2]

(i) Find the modulus and argument of $w$.

(ii) The complex number $z$ has modulus $R$ and argument $\theta$, where $-\frac{1}{3}\pi < \theta < \frac{1}{3}\pi$. State the modulus and argument of $wz$ and the modulus and argument of $\frac{z}{w}$. [4]

(iii) Hence explain why, in an Argand diagram, the points representing $z$, $wz$ and $\frac{z}{w}$ are the vertices of an equilateral triangle. [2]

(iv) In an Argand diagram, the vertices of an equilateral triangle lie on a circle with centre at the origin. One of the vertices represents the complex number $4 + 2i$. Find the complex numbers represented by the other two vertices. Give your answers in the form $x + iy$, where $x$ and $y$ are real and exact. [4]

May/June 2009

(i) Solve the equation $z^2 + (2\sqrt{3})iz - 4 = 0$, giving your answers in the form $x + iy$, where $x$ and $y$ are real. [3]

(ii) Sketch an Argand diagram showing the points representing the roots. [1]

(iii) Find the modulus and argument of each root. [3]

(iv) Show that the origin and the points representing the roots are the vertices of an equilateral triangle. [1]

May/June 2010/33

(a) The equation $2x^3 - x^2 + 2x + 12 = 0$ has one real root and two complex roots. Showing your working, verify that $1 + i\sqrt{3}$ is one of the complex roots. State the other complex root. [4]

(b) On a sketch of an Argand diagram, show the point representing the complex number $1 + i\sqrt{3}$. On the same diagram, shade the region whose points represent the complex numbers $z$ which satisfy both the inequalities $|z - 1 - i\sqrt{3}| \leq 1$ and $\arg z \leq \frac{1}{3}\pi$. [5]
7 The complex number \(-2 + i\) is denoted by \(u\).

(i) Given that \(u\) is a root of the equation \(x^3 - 11x - k = 0\), where \(k\) is real, find the value of \(k\). [3]

(ii) Write down the other complex root of this equation. [1]

(iii) Find the modulus and argument of \(u\). [2]

(iv) Sketch an Argand diagram showing the point representing \(u\). Shade the region whose points represent the complex numbers \(z\) satisfying both the inequalities

\[|z| < |z - 2| \text{ and } 0 < \arg(z - u) < \frac{1}{4}\pi.\] [4]

Oct/Nov 2009/32

7 The complex numbers \(-2 + i\) and \(3 + i\) are denoted by \(u\) and \(v\) respectively.

(i) Find, in the form \(x + iy\), the complex numbers

(a) \(u + v\), [1]

(b) \(\frac{u}{v}\), showing all your working. [3]

(ii) State the argument of \(\frac{u}{v}\). [1]

In an Argand diagram with origin \(O\), the points \(A, B\) and \(C\) represent the complex numbers \(u, v\) and \(u + v\) respectively.

(iii) Prove that angle \(AOB = \frac{3}{4}\pi\). [2]

(iv) State fully the geometrical relationship between the line segments \(OA\) and \(BC\). [2]

May/June 2010/31

7 The complex number \(2 + 2i\) is denoted by \(u\).

(i) Find the modulus and argument of \(u\). [2]

(ii) Sketch an Argand diagram showing the points representing the complex numbers \(1, i\) and \(u\). Shade the region whose points represent the complex numbers \(z\) which satisfy both the inequalities \(|z - 1| \leq |z - i|\) and \(|z - u| \leq 1\). [4]

(iii) Using your diagram, calculate the value of \(|z|\) for the point in this region for which \(\arg z\) is least. [3]

May/June 2010/32

8 The variable complex number \(z\) is given by

\[z = 1 + \cos 2\theta + i \sin 2\theta,\]

where \(\theta\) takes all values in the interval \(-\frac{1}{2}\pi < \theta < \frac{1}{2}\pi\).

(i) Show that the modulus of \(z\) is \(2\cos \theta\) and the argument of \(z\) is \(\theta\). [6]

(ii) Prove that the real part of \(\frac{1}{z}\) is constant. [3]